The structure of uniform B-spline curves with parameters

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Abstract

The shape-adjustable curve constructed by uniform B-spline basis function with parameter is an extension of uniform B-spline curve. In this paper, we study the relation between the uniform B-spline basis functions with parameter and the B-spline basis functions. Based on the degree elevation of B-spline, we extend the uniform B-spline basis functions with parameter to ones with multiple parameters. Examples show that the proposed basis functions provide more flexibility for curve design.

1. Introduction

B-spline curves are widely used in computer-aided geometric design and other associated fields, due to their excellent mathematical and algorithmic properties. In practical applications, designers often need manipulating a given curve into a desired shape. Traditionally, after we adopt a set of B-spline basis functions to construct curves, the shape of these curves can only be modified by adjusting control points. However, sometimes the designers may prefer to get different curves without changing the control points. To this end, we hope to construct shape-adjustable curves with fixed control points, namely, there exist appropriate flexibilities in the representations of the curves.

It is known that rational curves provide considerably more flexibilities in curve design than polynomial curves do, owing to the weights in their representations. We can also obtain curves with different shapes by choosing different weight values instead of moving control points. However, due to their fractional expressions, it is not easy to calculate the derivatives and integrals of the rational curves. Moreover, it is difficult to choose appropriate weight values to obtain a curve with desired shape. Fortunately, shape-adjustable curves can also be constructed by basis functions containing independent parameter, i.e. shape parameter. The earlier research using shape parameter to handle the curve shape design problem can be traced back to the paper of Barsky [1], in which the so called *Beta* spline and the concept of shape parameter were proposed. So far, many curves with single shape parameter have been developed, such as uniform B-spline polynomial curves with parameter, Bézier curves with parameter, C-Bézier curves, trigonometric/hyperbolic polynomial uniform B-spline with parameter and so on [2–10]. The curves constructed by these basis functions have many good properties similar to the corresponding ordinary (without parameters) ones.

Among the studies mentioned above, there are two literatures focusing on the construction of uniform B-spline curves with shape parameter. Han and Liu [2] proposed an extension of cubic uniform B-spline curves, where one shape parameter is introduced. And the uniform B-spline curves with a parameter of arbitrary degree (abbreviated as UBP basis functions herein) were defined by an integral recurrence formula in Ref. [6]. It can be easily verified that the basis functions proposed in Ref. [2] are the 4th order UBP basis functions proposed in Ref. [6].

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What we are interested in this study is the structure of UBP basis functions. We observed that, with the same order, the degree of UBP basis function was higher than that of B-spline basis functions. This difference guarantees the flexibility of the UBP basis functions. Furthermore, we found that UBP basis functions could be represented as linear combinations of B-spline basis functions (with a refined knot sequence), where the coefficients are expressed by the shape parameter. By comparing these combinations with the degree elevation formula of the B-spline basis functions, we are inspired to construct uniform B-spline basis functions with multiple parameters. These basis functions include the UBP basis functions, and share most nice properties with the UBP basis functions. The number of the shape parameters increases with the order of the basis functions.

2. Preliminary

2.1. B-spline basis functions

As the basic theory of B-spline is well known, we only give a brief summary of the concepts and notations. More details can be seen in Refs. [11–14].

The non-decreasing sequence of real numbers $T = \{t_0, \dots, t_n\}, t_i \leqslant t_{i+1}, i = 0, \dots, n-1$ is called the knot vector, where t_i is the knot. For $i = 0, 1, \dots, k$, the kth order ($(k-1)$th degree) B-spline basis functions $\tilde{N}_{i,k}(t)$ (or abbreviated as $\tilde{N}_{i,k}$) associated with knot vector T are defined as

$$\tilde{N}_{i,1}(t) = \begin{cases} 1 & t_i \leqslant t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{N}_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i}\tilde{N}_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}}\tilde{N}_{i+1,k-1}(t)$$

If the knots are equispaced $(t_i - t_{i-1} = \delta, i = 1, \dots, n)$, the B-spline basis functions will be assumed to be uniform, which are consistent under shifts $\tilde{N}_{i+1,k}(t) = \tilde{N}_{i,k}(t - \delta)$. By increasing the multiplicity of each knot of T by one, we obtain a refined knot sequence, and denote it by T^*. The kth order B-spline basis functions defined over T^* will be denoted as $\tilde{N}^*_{i,k}(t)$ or $\tilde{N}^*_{i,k}$. By degree elevation, a kth order B-spline basis function associated with T can be presented as linear combinations of the $(k+1)$th order B-spline basis functions associated with T^*. Namely,

$$\tilde{N}_{i,k}(t) = \sum_{j=p}^{q} a_{j,k} N^*_{i,k+1}(t) \tag{1}$$

where the support of $N^*_{i,k+1}(t)$,$j = p, \dots, q$ is included in the support of $\tilde{N}_{i,k}(t)$, and coefficients $a_{j,k}$, $j = p, \dots, q$ are determined by T^* [12].

2.2. Uniform B-spline basis functions with parameter

The kth order UBP basis functions $\bar{N}_{i,k}(t)$, $i = 0, \dots, k$ proposed in Ref. [6] are recursively defined as

$$\bar{N}_{0,2}(t) = \begin{cases} \frac{3}{2}\lambda t^2 + (1-\lambda)t & 0 \leqslant t < 1 \\ \frac{3}{2}\lambda(2-t)^2 + (1-\lambda)(2-t) & 1 \leqslant t < 2 \\ 0 & \text{other} \end{cases}$$

and

$$\bar{N}_{0,k}(t) = \int_{t-1}^{t} \bar{N}_{0,k-1}(t), \bar{N}_{i,k}(t) = \bar{N}_{0,k}(t-i) \tag{2}$$

When λ equals to 0, the UBP basis functions are B-spline basis functions. UBP basis functions share most properties with the uniform B-spline basis functions, such as non-negativity, local support, partition of unity, order of continuity (the kth order UBP basis functions are $(k-1)$ times continuously differentiable), linear independence and symmetry $(\bar{N}_{0,k}(t) = \bar{N}_{0,k}(k-t))$. Besides, their derivatives satisfy

$$\bar{N}'_{0,k}(t) = \bar{N}_{0,k-1}(t) - \bar{N}_{1,k-1}(t) = \bar{N}_{0,k}(t) - \bar{N}_{0,k-1}(t-1) \tag{3}$$

3. Structure of uniform B-spline basis functions with parameter

In this section, we study the structure of the UBP basis functions. We have the theorem as follows.

Theorem 1. *UBP basis functions $\bar{N}_{i,k}$ defined over T and B-splines basis functions $N^*_{i,k+1}$ defined over $T^*$are related by*

$$\bar{N}_{0,k} = \sum_{i=0}^{k} \lambda_{i,k} N^*_{i,k+1} \text{ and } \bar{N}_{i,k}(t) = \bar{N}_{0,k}(t-i) \tag{4}$$

where the coefficients $\lambda_{i,k}$ are given by

$$\lambda_{0,3} = \lambda_{3,3} = \frac{1-\lambda}{6}, \lambda_{1,3} = \lambda_{2,3} = \frac{5+\lambda}{6} \tag{5}$$

$$(\lambda_{0,k}, \dots, \lambda_{k,k}) = (\lambda_{0,k-1}, \dots, \lambda_{k-1,k-1}) \begin{pmatrix} \frac{\lfloor \frac{k}{2} \rfloor}{k} & \frac{\lfloor \frac{k}{2} \rfloor}{k} & 0 & 0 & \dots & 0 \\ 0 & \frac{\lceil \frac{k}{2} \rceil}{k} & \frac{\lceil \frac{k}{2} \rceil}{k} & 0 & \dots & 0 \\ \vdots & 0 & \frac{\lfloor \frac{k}{2} \rfloor}{k} & & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & & \dots & \frac{\lfloor \frac{k}{2} \rfloor}{k} \end{pmatrix}_{k \times k+1} \tag{6}$$

Proof. We prove this theorem by induction method. The theorem can be easily verified in the case of $k = 3$. Now assume that this theorem holds in the case of $k = l - 1$, i.e.

$$\bar{N}_{0,l-1} = \sum_{i=0}^{l-1} \lambda_{i,l-1} N^*_{i,l}$$

where $\lambda_{i,l-1}$ are parameters given by Eqs. (5) and (6). From Eq. (3) and

$$N^*_{i,l}(s-1) = N^*_{i+2,l}(s),$$

$${N^*_{i,l+1}}' = \frac{1}{t_{i+l} - t_i} N^*_{i,l} - \frac{1}{t_{i+l+1} - t_{i+1}} N^*_{i+1,l}$$

it follows that

$$\begin{aligned}\bar{N}_{0,l}(t) &= \int_{-\infty}^{t} \bar{N}_{0,l-1}(s) - \bar{N}_{0,l-1}(s-1)\mathrm{d}s \\ &= \int_{-\infty}^{t} \sum_{i=0}^{l-1} \lambda_{i,l-1} N^*_{i,l}(s) - \sum_{i=0}^{l-1} \lambda_{i,l-1} N^*_{i,l}(s-1)\mathrm{d}s \\ &= \int_{-\infty}^{t} \sum_{i=0}^{l-1} \lambda_{i,l-1} N^*_{i,l}(s) - \sum_{i=0}^{l-1} \lambda_{i,l-1} N^*_{i+2,l}(s)\mathrm{d}s \\ &= \sum\nolimits_{i=0}^{l-1} \lambda_{i,l-1}\delta_{i,l} \int_{-\infty}^{t} (N^*_{i,l+1}(s)' + N^*_{i+1,l+1}(s)')\mathrm{d}s \\ &= \sum\nolimits_{i=0}^{l-1} \lambda_{i,l-1}\delta_{i,l} (N^*_{i,l+1}(t) + N^*_{i+1,l+1}(t)) \\ &= \sum\nolimits_{i=0}^{l} \lambda_{i,l} N^*_{i,l+1}(t)\end{aligned}$$

where

$$\lambda_{i,l} = \lambda_{i-1,l-1}\delta_{i-1,l} + \lambda_{i,l-1}\delta_{i,l}$$

and

$$\delta_{i,l} = \begin{cases} \frac{t_{i+l}-t_i}{l} & i = 0, \ldots, l-1 \\ 0 & i = -1, l \end{cases} = \begin{cases} \frac{1}{2} & l \text{ is even}, i = 0, \ldots, l \\ \frac{l-1}{2l} & l \text{ is odd}, i = 0, 2, \ldots, l-1 \\ \frac{l+1}{2l} & l \text{ is odd}, i = 1, 3, \ldots, l \end{cases}$$

Hence, the theorem holds in the case of $k = l$, which completes the proof.

Theorem 1 illustrates that a kth order UBP basis function defined over T are polynomials of degree k, which can be presented as a linear combination of the $(k+1)$th order B-spline basis functions defined over T^*. And the combination form is similar to the B-spline degree elevation formula Eq. (2), except that the coefficients here are variables with respect to λ. Additionally, the coefficients $\lambda_{i,k}$ have the properties as follows.

Remark 1. The coefficients in Eq. (6) satisfy

$$0 \leqslant \lambda_{i,k} = \lambda_{k-i,k} \leqslant 1 \text{ for } i = 0, \ldots, \left\lceil \frac{k}{2} \right\rceil$$

and

$$\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \lambda_{2i,k} = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \lambda_{2i+1,k} = \delta_{0,k} + \delta_{1,k} = 1$$

For any polynomial, one degree of freedom can be introduced to its expression by raising its degree by one. Note that B-spline basis functions are piecewise polynomials. Therefore, degree elevation method can theoretically introduce multiple degrees of freedom into the expressions of B-spline curves. However, Theorem 1 reveals that, the UBP basis functions are the revised B-spline basis functions, whose expressions are modified by inserting only one parameter by the degree elevation of B-spline. Thus, we will study if there are other more general uniform B-spline basis functions with multiple parameters in the next section.

4. General B-spline basis functions with multiple parameters

In this section, we will propose the new uniform B-spline basis functions with parameters by the degree elevation of B-spline, in which the possibly greatest degree of the freedom is exploited. The obtained basis functions will include UBP basis functions and have all the properties mentioned in Section 2.2. We call the basis functions proposed here the general B-spline basis functions with multiple parameters, abbreviated as GUBP basis functions. The kth order GUBP will be denoted by $N_{i,k}(t)$ or $N_{i,k}$. The main idea to construct the kth order GUBP is replacing all the constant coefficients $a_{i,k}$ in the degree elevation formula Eq. (2) with independent parameters, and adding appropriate restrictions to these independent parameters to guarantee the properties mentioned in subsection 2.2.

In the rest of this section, we will go into the details of the construction process.

Note that when $i = 0$, p and q in Eq. (2) equal to 0 and k, respectively. By substituting independent parameters $\mu_{j,k}$ for constant coefficients $a_{j,k}$ in Eq. (2), we obtain a rough kth order GUBP basis function as

$$N_{0,k} = \sum_{i=0}^{k} \mu_{i,k} N^*_{i,k+1} \tag{7}$$

Other basis functions are defined by shifts

$$N_{i,k}(t) = N_{0,k}(t-i) \tag{8}$$

The rest work is to restrict $\mu_{j,k}$ appropriately to ensure the properties of GUBP basis functions mentioned above. Since GUBP basis functions are consistent under the shifts Eq. (8), we only need to deal with $N_{0,k}$ in some cases in the following discussion. First, we require that all the parameters are non-negative, i.e.

$$\mu_{i,k} \geqslant 0 \text{ for } i = 0, \ldots, k$$

It follows that all GUBP basis functions hold the properties of non-negative, local support and order of continuity.

Now we deal with the property of symmetry based on the observation of the images of $N^*_{i,k}$ (see Fig. 1). Note that all the knots of T^* are multiple knots of multiplicity two and then $N^*_{i,k}$ satisfy the relations

$$\begin{cases} N^*_{2i,k}(t) = N^*_{0,k}(t-i) \\ N^*_{2i+1,k}(t) = N^*_{1,k}(t-i) \end{cases}$$

We classify $N^*_{i,k}$ into two kinds, e-kind and o-kind, according to subscript i being even or odd. Note that $N^*_{i,k+1}$ and $N^*_{k-i,k+1}$ are symmetrical with respect to axis $t = \lfloor \frac{k}{2} \rfloor$. Hence, in order to keep the property of symmetry of $N_{0,k}$, the following restriction on the parameters $\mu_{j,k}$ is required:

$$\mu_{i,k} = \mu_{k-i,k}, \text{ for } i = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$$

We still have to tackle the property of partition of unity. Note that the support interval of an e-kind (o-kind) basis function is $\lfloor \frac{k+1}{2} \rfloor (\lceil \frac{k+1}{2} \rceil)$, which implies that

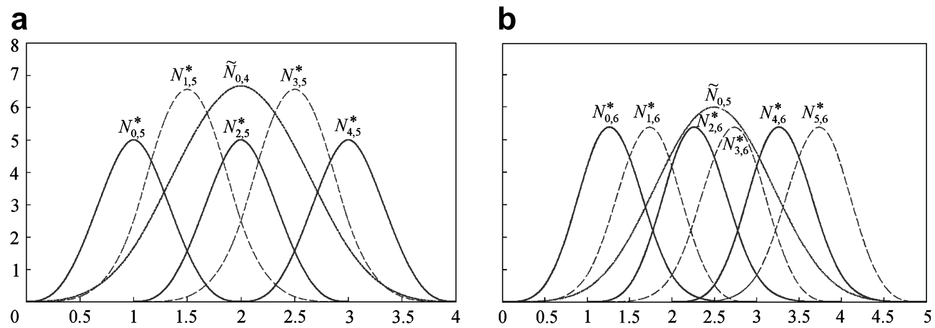

Fig. 1. The images of B-spline basis functions defined over T and T^*. Dot lines denote $\tilde{N}_{0,k}$ defined over T, solid lines and dash lines denote e-kind and o-kind of $N^*_{i,k+1}$ defined over T^*, respectively. (a) $k = 4$; (b) $k = 5$.

each e-kind (o-kind) basis function is simultaneously involved in the representation of $\lfloor\frac{k+1}{2}\rfloor(\lceil\frac{k+1}{2}\rceil)$ GUBP basis functions. For example, when k is even, Eq. (7) implies that $N^*_{k,k+1}$, associated with parameter $\mu_{k,k}$, are included in the expression of $N_{0,k}$ (see Fig. 1). Simultaneously, we can infer easily from Eq. (7) that $N^*_{k,k+1}$ engages in the expressions of $N_{1,k}, N_{2,k}, \ldots, N_{\frac{k}{2}-1,k}$ and $N_{\frac{k}{2},k}$, with corresponding parameters $\mu_{k-2,k}, \mu_{k-4,k}, \ldots, \mu_{1,k}$ and $\mu_{0,k}$. Thus, in order to guarantee the property of partition of unity of GUBP basis functions, all parameters before the same basis function $N^*_{i,k+1}$ must be summed to 1, i.e.

$$\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\mu_{2i} = 1 \text{ and } \sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\mu_{2i+1} = 1$$

Finally, since B-spline basis functions are linearly independent, the linear independence of GUBP basis functions becomes a natural result.

Now, we give the formal definition of GUBP basis functions as follows.

Definition. The kth ($k \geqslant 3$) order general uniform B-spline basis functions with parameters associated with knot vector T are defined as

$$N_{0,k} = \sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\mu_{i,k}(N^*_{i,k+1} + N^*_{k-i,k+1}) + \begin{cases}\mu_{\frac{k}{2},k}N^*_{\frac{k}{2},k+1} & k \text{ is even}\\ 0 & k \text{ is } odd\end{cases}$$

where $N^*_{i,k+1}$ are B-spline basis functions associated with the refined knot vector T^* and for integer $n \geqslant 1$, $0 \leqslant \mu_{i,k} \leqslant 1$ satisfy

$$\begin{cases}\sum_{i=0}^{n-1}\mu_{2i,k} + \frac{1}{2}\mu_{2n,k} = \frac{1}{2}, \sum_{i=0}^{n-1}\mu_{2i+1,k} = \frac{1}{2} & k = 4n\\ \sum_{i=0}^{n}\mu_{2i,k} = \frac{1}{2}, \sum_{i=0}^{n-1}\mu_{2i+1,k} + \frac{1}{2}\mu_{2n+1,k} = \frac{1}{2} & k = 4n+2\\ \sum_{i=0}^{\frac{k-1}{2}}\mu_{i,k} = 1 & k = 4n-1, 4n+1\end{cases} \tag{9}$$

Remark 2. Eq. (9) implies that the parameters contained in the kth order GUBP basis functions have freedom of degree $\lfloor\frac{k-1}{2}\rfloor$.

Theorem 1 also implies that we can convert GUBP basis functions to UBP basis functions by setting $\mu_{i,k} = \lambda_{i,k}$ for $i = 0, \ldots, \lceil\frac{k}{2}\rceil$. In other words, GUBP basis functions subsume the UBP basis functions as special cases. Furthermore, it can be checked that the rules of the parameters $\lambda_{i,k}$ presented in Remark 1 satisfy restrictions Eq. (9), which

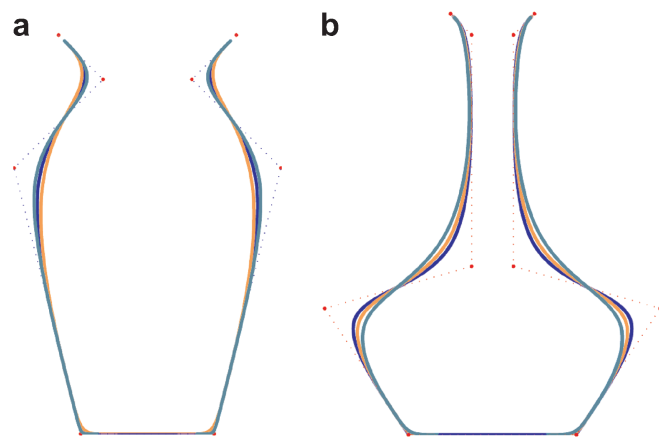

Fig. 2. Two digital ceramic vessels.

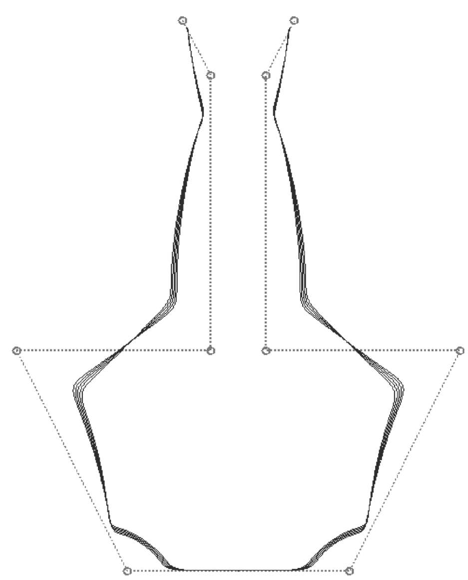

Fig. 3. The 5th order UBP curves with $\lambda = 0.25, 0.5, 0.75, 1$.

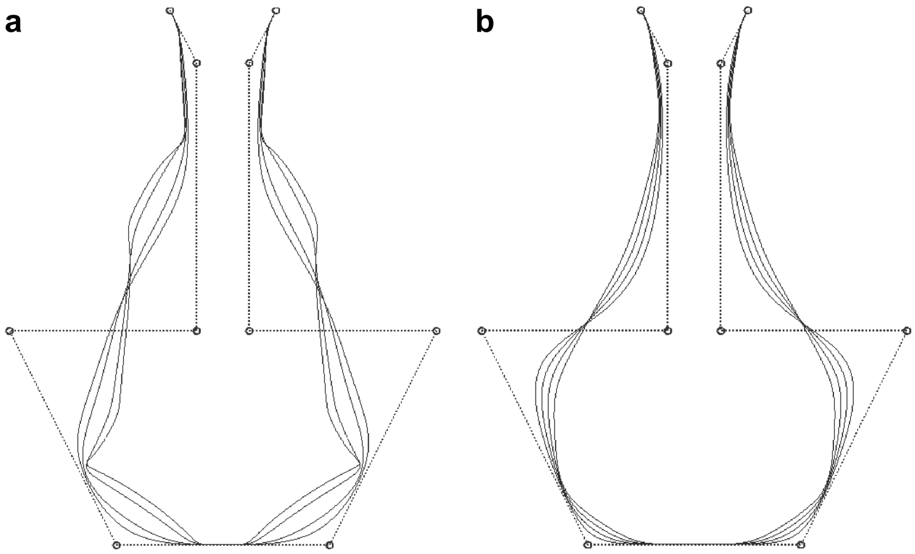

Fig. 4. The 5th order GUBP curves. (a) $\mu_{0,5} = \mu_{2,5} = 0.25, 0.3, 0.4, 0.45$; (b) $\mu_{0,5} = 0.3$ and $\mu_{2,5} = 0.5, 0.55, 0.6, 0.65$.

endows the UBP basis functions with the aforementioned properties.

5. Examples

In this section, we give some examples of the application of vessel styling design. We call the curves constructed by GUBP basis functions and UBP basis functions GUBP curves and UBP curves, respectively. According to Remark 2, GUBP curves with order higher than 4 contained more than one shape parameters. For example, the 5th order GUBP curves have two freedom parameters, since their three parameters $\mu_{0,5}, \mu_{1,5}, \mu_{2,5}$ are under the restriction $\mu_{0,5} + \mu_{1,5} + \mu_{2,5} = 1$. Hence, we can choose, e.g. $\mu_{0,5}$ and $\mu_{2,5}$, as shape parameters, then $\mu_{1,5}$ can be decided by $\mu_{1,5} = 1 - \mu_{0,5} - \mu_{2,5}$.

Two vessels constructed by the 4th order GUBP basis functions under fixed control polygons are shown in Fig. 2. Curves with different colors are obtained by choosing different shape parameters. The fifth order UBP curves and GUBP curves under the same control polygon are shown in Fig. 3 and Fig. 4, respectively. The UBP curves displayed in Fig. 3 are curves with shape parameter λ being 0.25, 0.5, 0.75 and 1. The GUBP curves shown in Fig. 4a are obtained by setting $\mu_{0,5} = \mu_{2,5} = 0.25, 0.3, 0.4, 0.45$. By fixing $\mu_{0,5} = 0.3$ and assigning 0.5, 0.55, 0.6, 0.65 to $\mu_{2,5}$, we obtain the 5th order GUBP curves in Fig. 4b. We can see that the UBP curve shapes shown in Fig. 3 are lack of variety. On the contrary, the GUBP curves shown in Fig. 4 have more changeful shapes. In particular, the shapes shown in Fig. 4b are more acceptable as a vase shape. These examples show that the GUBP curves provide more flexibility than the UBP curves.

6. Conclusion

We have proposed the GUBP basis functions based on the degree elevation of B-spline, which are the extensions of the known UBP basis functions. Compared with the existing one, the GUBP basis functions (with order higher than four) are advantageous in having more than one shape parameter. Additionally, the number of the shape parameters will increase with the order of the GUBP basis functions. Examples show that curves constructed by the basis functions proposed in this paper have profuse shapes.

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